Math 171 Proficiency Packet on Integers

Section 1: Integers

For many of man's purposes the set of whole numbers $W = \{0, 1, 2, ...\}$ is inadequate. It became necessary to invent negative numbers and extend the set of whole numbers to the set of integers:

$$I = {..., -3, -2, -1, 0, 1, 2, 3, ...}.$$

Integers are used to record debits and credits, profits and losses, changes in the stock market, degrees above zero and degrees below zero in measuring temperature, yards lost and yards gained in a football game, and so on.

A number line can be used to represent the integers. Start out by choosing an arbitrary point on the number line to represent 0. Once we know unit distance we can label successive points on the right side of 0 with **positive integers** (the natural or counting numbers) and points on the left with successive **negative integers** (-1, -2, -3, ...).

This number line model of the integers is useful in showing their relative sizes (smaller integers lie to the left of larger integers) and for illustrating that the nonnegative integers can be identified with the whole numbers. Note that if \mathbf{a} and \mathbf{b} are integers, and if \mathbf{a} is larger than \mathbf{b} , \mathbf{a} is to the right of \mathbf{b} on the number line. This is written symbolically as $\mathbf{b} < \mathbf{a}$ or $\mathbf{a} > \mathbf{b}$.

Example 1: Order the following integers from the smallest to largest. -1, -13, 5, 0, 8, -12

Solution: -13 < -12 < -1 < 0 < 5 < 8

Now You Try (Section 1.1)

Order the following integers from smallest to largest. It is helpful to visualize where these integers belong on a number line.

(Answers to **Now You Try** (Section 1.1) are found on page 18.)

Now let's look at **addition and subtraction** of the integers using the number line. To find -3 + -2 using a number line, picture this sum as the result of starting at the origin and moving two units to the left and then another 3 units again to the left. Doing this we end up at -5. So, -3 + -2 = -5.

Now You Try (Section 1.2)

Calculate the following using a number line.

(Answers to Now You Try (Section 1.2) are found on page 18.)

One way to look at subtraction is with a **missing addend interpretation**. When we do the subtraction 5 - 2 we are really looking for the number we would add to 2 to get 5. Therefore, 5 - 2 = 3 since 3 + 2 = 5.

Example 1: Compute -5 - -3.

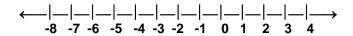
Solution: Using the **missing addend interpretation**, we must find the solution to $\Box + -3 = -5$. A number line may be useful in visualizing this equation

We are at -3 and are being asked, "what has to be added to -3 so that we end up at -5?" Since we have to move to the left 2 units, -2 + -3 = -5. Therefore, -5 - -3 = -2.

Now You Try (Section 1.3)

Calculate the following. Use a number line to check your solutions.

1) -6 - -5 = since + -5 = -6.



2) -3 - 8 =_____ since + -8 = -3.

3) -6 - 6 =_____since + 6 = -6.

(Answers to **Now You Try** (Section 1.3) are found on page 19.)

Section 2: Addition of Integers

Before we discuss addition of integers, we need to define the absolute value of an number.

<u>Absolute Value:</u> The distance from zero to a number on the number line. The absolute value will either be <u>positive</u> or <u>zero</u>. The symbol for absolute value is two vertical bars $| \ |$.

Example 1: Evaluate:

- a) |-5|
- b) |8|
- c) |0|
- d) |7 + 3 |

Solution:

- a) Since -5 is 5 units away from zero the absolute value of -5 equals 5. (|-5| = 5)
- b) Since 8 is 8 units away from zero, the absolute value of 8 equals 8. (|8| = 8)
- c) |0| = 0
- d) |7+3| = |10| = 10

Now You Try (Section 2.1)

Evaluate:

- 1) |-7|
- 2) |5-5|
- 3) |24|

(Answers to Now You Try (Section 2.1) are found on page 19.)

Now that you understand the absolute value, we can discuss the rules for adding integers.

Adding Two Integers

Example 1:

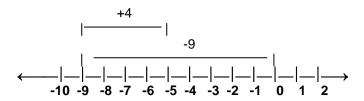
- a) 7 + 9 = 16
- b) -5 + -3 = |-5| + |-3| = 5 + 3 = 8 since they are both negative, the answer is -8.

We can also add, using the number line.

Example 2:

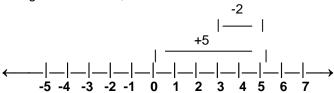
a) -9+4=|-9|-|4|=9-4=5, since -9 has the larger absolute value, the answer is -5.

Using a number line,



5 + -2 = |5| - |-2| = 5 - 2 = 3, since 5 has the larger absolute value, the answer b) is 3.

Using a number line,



If you have more than 2 integers to add together, you can either;

- (1) Work left to right, adding 2 at a time or,
- (2) Add the positives, add the negatives and then add those 2 answers.

Example 3: Add: -7 + 10 + 12 + -8 + 3

Using method (1)

Using method (2)

-7 + 10 + 12 + -8 + 3add the positives: 10 + 12 + 3 = 25

-7 + -8 = -153 + 12 + -8 + 3add the negatives:

add those 2 answers: 25 + -15 = 10. <u>15 + -8</u> + 3

7 + 3

10 .

Now You Try (Section 2.2)

Calculate:

1) 18 + -52) -9 + -12

27 + -24 + 12 + -6 -23 + 103)

-39 + -11 + -8 + 23 5)

(Answers to Now You Try (Section 2.2) are found on page 19.)

Section 3: Subtraction of Integers

When we subtract integers, we add the opposite.

Let a and b be any integers, then a - b = a + (-b)

Use the following steps when subtracting 2 integers:

- 1) Rewrite the 1st integer as it is given.
- 2) Change the subtraction sign to an addition sign.
- 3) Write the opposite of the 2nd number.
- 4) Use the rules for addition to find the answer.

Example 1:

a)
$$12 - 27 = 12 + (-27) = -15$$

∠ (different signs - subtract and keep the sign of the larger absolute value)

b)
$$-12 - 27 = -12 + (-27) = -39$$

∠ (same signs - add and keep the common sign)

c)
$$-12 - (-27) = -12 + (+27) = 15$$

∠ (different signs - subtract and keep the sign of the larger absolute value)

d)
$$12 - (-27) = 12 + (+27) = 39$$

∠ (same signs - add and keep the common sign)

If you have more than 2 integers to subtract, change all the subtractions to additions of the opposite and then use the rules for addition.

Example 2:

$$= -5 + 7 + -18 + 6 = -23 + 13 = -10$$
.

Now You Try (Section 3)

Calculate:

(Answers to Now You Try (Section 3) are found on page 19.)

Section 4: Multiplication and Division of Integers

We have 2 different cases when multiplying or dividing 2 integers.

<u>Case 1:</u> <u>Same Signs</u> → MULTIPLY or DIVIDE as usual, answer will be <u>POSITIVE</u>.

Example 1:

a)
$$7 \cdot 9 = 63$$

c)
$$\frac{35}{5} = 7$$

b)
$$-5 \cdot -3 = 15$$

$$\frac{-48}{-6} = 8$$

<u>Case 2:</u> <u>Different Signs</u> → MULTIPLY or DIVIDE as usual, answer will be <u>NEGATIVE</u>.

Example 2:

a)
$$-7 \cdot 9 = -63$$

d)
$$\frac{48}{-6} = -8$$

b)
$$-5 \cdot 3 = -15$$

e)
$$\frac{-15}{0}$$
 = undefined

(Recall that you cannot divide by 0.)

c)
$$\frac{-35}{5} = -7$$

(Notice when multiplying or dividing two integers, it doesn't matter which number is larger, if one number is negative and the other is positive, the answer is negative.)

If you have more than 2 integers to multiply together you can either;

- (1) Work left to right, multiplying 2 at a time, or
- Ignore the signs and multiply all the numbers, then count the # of negatives if there are an even # of negatives, the answer is positive; if there are an odd # of negatives, the answer is negative.

Example 3:

$$(-3)(-2)(-6)(-1)(4)$$

Using method (1)

<u>(-3)(-2)(-6)(-1)(4)</u>

$$(-3)(-2)(-6)(-1)(4)$$

<u>6(-6)</u>(-1)(4)

<u>-36(-1)(4)</u>

$$(3)(2)(6)(1)(4) = 6 \cdot 6 \cdot 4 = 36 \cdot 4 = 144$$
.

.

Since there is an even number of negatives, the answer is positive.

(36)(4)

144.

Now You Try (Section 4)

Calculate:

2)
$$\frac{-78}{-2}$$

4)
$$\frac{96}{-12}$$

$$6) \qquad \frac{-8 \cdot 6}{0}$$

(Answers to Now You Try (Section 4) are found on page 20.)

Section 5: Order of Operations

You know from previous courses that if two quantities are added, it does not make a difference which quantity is added to which. For example, 5+6=6+5. This you will recall is the **commutative property of addition**. Even if you added more than two quantities it does not matter the order they are added. The same is true for multiplication. If you had to multiply $2 \cdot 4 \cdot 8$, you would always get 64 not matter what order the numbers were multiplied. What happens though when the operations are not all only addition or multiplication? For example, compute $2+6\cdot 5$. You might do it from left to right and get 40 and a friend of yours might first multiply 6 and 5 and then add 2 to get an answer of 32. We cannot have this inconsistency, so it is important to have a set of rules (**order of operations**) to prevent different answers from occurring. The correct answer is 32. The order of operations tells us to multiply before we add.

The following is a summary of the order of operations.

ORDER OF OPERATIONS

If **no** grouping symbols are present do the following:

- 1. Evaluate all powers, working from left to right.
- 2. Do any multiplications and divisions in the **order** in which they occur, working from left to right.
- 3. Do any additions and subtractions in the **order** in which they occur, working from left to right.

If grouping symbols are present, do the following:

- 1. First use the above steps within each pair of parentheses.
- 2. If the expression includes nested grouping symbols, evaluate the expression in the innermost set of parentheses first.

Grouping symbols may be (), [], or {}.

Note: Some people use the acronym "Please Excuse My Dear Aunt Sally" to remember the order of operations. Here the P stands for parentheses, E for exponent, M for multiplication, D for division, A for addition, and S for subtraction. This acronym can be very misleading for people because it can be misinterpreted to mean that multiplication must be performed before division and addition before subtraction. Remember, you do multiplication and division the order in which they appear, if a division appears before a multiplication you do division first. Similarly, if a subtraction appears before an addition, you do subtraction first.

Example 1: Evaluate $7 + 3 \cdot 2 - 18 \div 6$ without using a calculator.

Solution: No parentheses or exponents, so do multiplications and divisions from left to right in the order they appear.

$$7 + 3 \cdot 2 - 18 \div 6 = 7 + 6 - 3$$
 $\uparrow \uparrow$
 $3 \cdot 2 \cdot 18 \div 6$
 $= 13 - 3$

Working left to right, add 7 and 6.

 $= 10$

Subtract 3 from 13.

Example 2: Evaluate 6 + 3(8 - 3) without using a calculator.

Solution: According to the order of operations, we have to do what is inside the parentheses first:

$$6 + 3(8 - 3) = 6 + 3(5)$$
 Do inside the parentheses first.
 $= 6 + 15$ Do multiplication before addition.
 $= 21$ Add 6 and 15.

Example 3: Evaluate 6 - 4[8 + 3(7 - 3)] without using a calculator.

Solution: This expression has nested parentheses where the [] are used in the same way as parentheses. In a case like this we move to the innermost set of parentheses first and begin simplifying:

$$6 - 4[8 + 3(7 - 3)] = 6 - 4[8 + 3(4)]$$
 (7-3); innermost grouping first.
$$= 6 - 4[8 + 12]$$
 3(4); Multiplication before addition.
$$= 6 - 4[20]$$
 (8 + 12); Stay within [] until you get a single number.
$$= 6 - 80$$
 4[20]; Multiplication before subtraction.
$$= -74$$
 Subtract 80 from 6.

Example 4: Evaluate $18 \div 3 \cdot 2 + 7 \cdot 2 \cdot 4$ without using a calculator.

Solution: There are no parentheses or exponents, se we go from left to right and do all multiplications and divisions in the order they appear. In this case a division comes before a multiplication, so the division is done first.

$$18 \div 3 \cdot 2 + 7 \cdot 2 \cdot 4 = 6 \cdot 2 + 7 \cdot 2 \cdot 4$$
$$= 12 + 7 \cdot 8$$
$$= 19 \cdot 8$$
$$= 11$$

Example 5: Evaluate $9 \cdot 2^3 + 36 \div 3^2 - 8$ without using a calculator.

Solution: We do exponents first.

$$9 \cdot 2^3 + 36 \div 3^2 - 8 = 9 \cdot 8 + 36 \div 9 - 8$$
Now do multiplication and divisions from left to right.
$$= 72 + 4 - 8$$
Now do additions and subtractions from left to right.
$$= 76 - 8$$

$$= 68$$

When computing an expression such as $\frac{2\cdot 5+8}{6-3}$, the fraction bar has the same function as grouping symbols. Any operations that appear above or below a fraction bar should be completed first.

Example 6: Compute $\frac{2.5 + 8}{6 - 3}$.

Solution: We have to compute $2 \cdot 5 + 8$ and 6 - 3 before we divide.

$$\frac{2.5 + 8}{6 - 3} = \frac{18}{3} = 6$$

In order to compute this example on a calculator or a computer, you would have to write this problem in a horizontal line.

So $\frac{2.5+8}{6-3}$ becomes $(2.5+8) \div (6-3)$ or (2.5+8)/(6-3).

Now You Try (Section 5)

Evaluate the following without using a calculator:

- 1) 3 + 4(3 + 12)
- 2) $12 \div 6 \cdot 2 7(3 2)$
- 3) 4[6 4(2 7)]
- 4) 8·2 4 10 4·3

(Answers to Now You Try (Section 5) are found on page 20.)

Section 6: Exponents

You probably have some experience with exponents from previous courses. Just as multiplication can be thought of as a shorthand notation for repeated addition, $3 \cdot 4 = 4 + 4 + 4$, exponents are a shorthand notation for repeated multiplication, $2^3 = 2 \cdot 2 \cdot 2$. So **exponential notation** is just a shorthand system, but like any special notation you have to learn to read and use it correctly.

Exponential Notation	Let x be a real number and n be a positive integer. The product of n of the x's
	(x)(x)(x)(x) n times
	is called
	"x to the nth power"
	or
	"x to the nth"
	or
	"x raised to the n."
	We call n the exponent or "power" and x the base , and write (x)(x)(x)(x) simply as x ⁿ . n times

When dealing with exponential notation it is very important to determine what the base of an exponent is. **As a general rule, the exponent applies to the symbol that precedes it.** For example, the symbol that precedes the exponent in 5^3 is 5, so the base for the exponent is 5. This means that 5 is used as a factor 3 times, $5^3 = (5)(5)(5)$. Now consider $(4x)^2$, here the symbol that preceded the exponent is parentheses. This tells us that everything inside the parentheses is the base, so here 4x will be used as a factor twice. In other words, $(4x)^2 = (4x)(4x)$. Remember that when parentheses are used, it is the entire quantity inside the parentheses that is repeated as the factor.

Now You Try (Section 6.1)

Name the base of each exponent.

1)
$$(-5)^2$$

3)
$$(x + y)^2$$

(Answers to Now You Try (Section 6.1) are found on page 20.)

Now let's look at some order of operation examples that contain exponents.

Example 1: Without using a calculator, evaluate $4 + 5\{3 - 3(12 \div 6 \cdot 3 - 5 + 6)\}^3$.

Solution:

$$4 + 5{3 - 3(12 \div 6 \cdot 3 - 5 + 6)}^{3}$$

$$= 4 + 5{3 - 3(2 \cdot 3 - 5 + 6)}^{3}$$

$$= 4 + 5{3 - 3(6 - 5 + 6)}^{3}$$

$$= 4 + 5{3 - 3(1 + 6)}^{3}$$

$$= 4 + 5{3 - 3(7)}^{3}$$

$$= 4 + 5{3 - 21}^{3}$$

$$= 4 + 5{3 - 21}^{3}$$

 $= 4 + 5 \cdot (-5832)$ = 4 + (-29160)= -29156

Evaluate the exponent. Multiply.

Add.

Evaluate $\frac{5(2^3-1)}{2-1}$. Example 2:

> Solution: According to the order of operations,

$$\frac{5(2^3-1)}{2-1} = \frac{5(8-1)}{2-1} = \frac{5\cdot7}{1} = \frac{35}{1} = 35$$

Now You Try (Section 6.2)

Evaluate the following without using a calculator.

1)
$$5 \cdot 3^4 + 16 \div 8 - 2^2$$

2)
$$4 + 3(5 - (12 \div 6 \cdot 4 - 4^2))^2$$

(Answers to Now You Try (Section 6.2) are found on page 20.)

Section 7: Long Division

Terms of Division

$$\frac{\text{quotient}}{\textit{divisor}} \frac{\text{dividend}}{\textit{dividend}}$$

remainder

The traditional algorithm for division is illustrated in the next two examples.

Example 1:

Solution:

Since 7(6) = 42, our first estimate of the number of sevens that can be subtracted from 47 is 6:

$$\frac{6}{7)476}$$

$$-42$$

7(6) = 42, place the 6 in the quotient above the tens column

47 - 42 = 5; then bring down the 6

Since 7(8) = 56, we write 8 in the quotient and multiply. The final work is shown on the next page.

$$\frac{68}{\sqrt{476}}$$

$$\frac{42}{56}$$

$$\frac{56}{\sqrt{60}} \leftarrow 7(8) = 56$$
, place the8 in the quotient above the ones column $\sqrt{60}$

Our result is $476 \div 7 = 68$, which we can check with the following equation:

Example 2: Divide: 33,997 ÷ 56

Solution: Since 50(6) = 300, our first estimate of the number of fifty-sixes that can be subtracted from 339 is 6.

$$\begin{array}{rcl}
 & 6 \\
 & 56 \overline{\smash)33997} \\
 & \underline{} & \underline{} & 56(6) = 336, \, place \, the6 \, in \, the \, quotient \, above \, the \, hundreds \, column} \\
 & \underline{} & 339 & 4 & 339 - 336 = 3, \, then \, bring \, down \, the \, 9
\end{array}$$

Since 59 cannot be divided into 39, we write a 0 in the quotient in the tens column.

$$\begin{array}{rcl}
 & 607 \\
56 \overline{\smash)33997} \\
 & \underline{397} \\
 & \underline{397} \\
 & \underline{56(7)} = 392, \, place \, the7 \, in \, the \, quotient \, above \, the \, ones \, column} \\
 & \underline{56(7)} = 392, \, place \, the7 \, in \, the \, quotient \, above \, the \, ones \, column} \\
 & \underline{56(7)} = 397 - 392 = 5, \, write \, 5 \, as \, the \, remainder.
\end{array}$$

Therefore, $33,997 \div 56 = 607 R 5$.

Check:
$$56(607) + 5 = 33992 + 5 = 33997 \sqrt{}$$

Now You Try (Section 7)
1) Divide 274 by 8.
2) Divide 1238 by 23.
2) Divide 1200 by 20.
(Answers to Now You Try (Section 7) are found on page 20.)
(

Exercises for Integers

Do all the exercises on separate paper showing all your work.

- 1. Draw number line diagrams that illustrate each computation and state the answer in each case.
 - a) 8 + (4)
- b) $^{-}7 + 9$
- c) ⁻5 5

- 2. Write each of these subtractions as an addition.
 - a) 15 8
- b) 5 (3)
- c) ⁻56 (⁻64)
- d) ⁻33 18
- 3. Perform the following computations.
 - a) ⁻12 14
- b) 75 (43)
- c) 23 (67)

d) 5 + 8

- e) ⁻⁸⁵ + 42
- f) -12 12

- g) ⁻³⁴ + 54
- h) ⁻³⁴ 54
- I) $15 + (^{-}30) + 18 + ^{-}9$

- j) 12 -(⁻21) 34 (⁻60)
- k) $25 54 + (\bar{3}3)$
- 4. List these numbers in increasing order from least to greatest.
 - 4, -3, 0, 1, -5, -56
- 5. Compute the following:
 - a) $18 \div -3$

b) -6 · -5

c) -6 + -14

d) -5 - -7

e) 72 · -2

f) -6(-14)

g) -48 ÷ 3

h) $0 \div 18$

i) -7 ÷ 0

j) (7)(-5)(2)(0)

k) (-5)(-2)(-4)(3)

- 6. True or false
 - a) The sum of a positive number and a negative number is negative.
 - b) The difference of two negative numbers is positive.
 - c) Dividing a positive integer by a negative integer, and multiplying the result by a negative integer gives a positive result.
- 7. Evaluate each of the following without using a calculator.

a)
$$12 - 4 \div 2$$

b)
$$10 - 4(20 \div 4 \cdot 5 - 7 + 5)$$

d)
$$3 - 6(3 \cdot 2 - 2)^2$$

e)
$$4 \cdot 5^3 - 20 \div 5 \cdot 4$$

f)
$$6^2 - 10 \div 2 \cdot 5^2$$

g)
$$5(3+4)^2$$

8. Using exponents, write the following in simpler form:

9. Name the base and the exponent in each of the following:

a)
$$(3x)^2$$

b)
$$4x^{3}$$

d)
$$(x + 6)^2$$

- 10. Find the quotient and remainder when
 - a) 26751 is divided by 22
 - b) 48233 is divided by 30

Answers to Now You Try

Section 1.1:

Section 1.2:

so,
$$-2 + -5 = -7$$

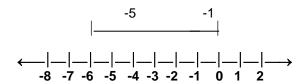
so,
$$-3 + -4 = -7$$

so,
$$-4 + 7 = 3$$

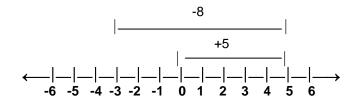
so,
$$5 + -4 = 1$$

Section 1.3:

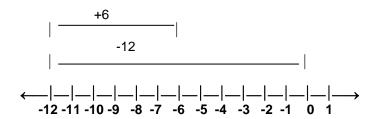
1) -6 - 5 = -1 since -1 + 5 = 6



2) 3 - 8 = 5 since 5 + 8 = 3



3) -6-6 = -12 since -12+6 = -6



Section 2.1:

- 1) 7
- 2) 0
- 3) 24

Section 2.2:

- 1) 13
- 2) -21
- 3) -13
- 4) 9
- 5) -35

Section 3:

- 1) -78
- 2) -26
- 3) 13

- 4) 45
- 5) 4
- 6) 6

Section 4:

6)
$$\frac{-48}{0}$$
 = undefined

Section 5:

Section 6.1:

Section 6.2:

Section 7:

1)
$$8)274$$

so
$$247 \div 8 = 34 R 2$$
.

2)
$$\frac{53}{23)1238}$$

$$\frac{115}{88}$$

$$\frac{69}{19}$$