Fractions and Ratios

Section 1: Introduction

Recall the following sets of numbers; the set of natural numbers, N = {1, 2, 3, …} the set of whole numbers, W = {0, 1, 2, 3, 4, ...}, and the set of integers I = { ..., -2, -1, 0, 1, 2, ... }. In this unit we extend the set of integers to a number system called the **rational numbers**. Rational numbers were developed because there are instances when whole numbers and integers cannot fully describe or quantify the situation. For example, how would you describe the portion of the large rectangle shaded in the following diagram?

xxxxxxxxxxxxxx	xxxxxxxxxxxxxx	
xxxxxxxxxxxxxx		

You can see that we have to extend the integers in order to handle a situation like this.

Definition: A **fraction** is a number of the form $\frac{a}{b}$, where a and b are integers and b $\neq$ 0.

Some examples of fractions are

$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{7}$	$\frac{8}{5}$
Two-thirds	Three-fifths	Six-sevenths	Eight-fifths

Definition: For the fraction $\frac{a}{b}$, a and b are called the **terms** of the fraction. More specifically, a is called the **numerator** and b is called the **denominator**.

Example 1 : The terms of the fraction $\frac{2}{3}$ are 2 and 3. The 2 is called the **numerator**, and the 3 is called the **denominator**.

Example 2: An integer such as 5 may also be put in fraction form, since 5 can be written as $\frac{5}{1}$. In this case, 5 is the **numerator**, and 1 is the **denominator**.

Definition: A **proper fraction** is a fraction in which the numerator is less than the denominator. If the numerator is greater than or equal to the denominator, the fraction is called an **improper fraction**.

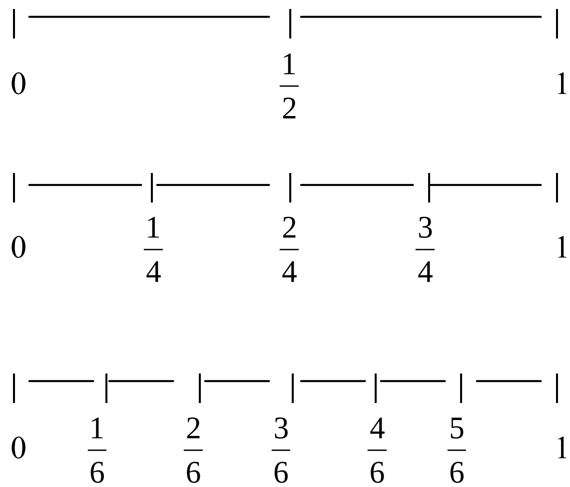
In example 1, $\frac{2}{3}$ is a **proper** fraction and $\frac{5}{1}$ is an **improper** fraction in example 2.

Section 2: Equivalent Fractions

Definition: When two fractions represent the same amount with respect to the same "whole", the two fractions are said to be **equivalent.**

Example 3 : $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \ldots$

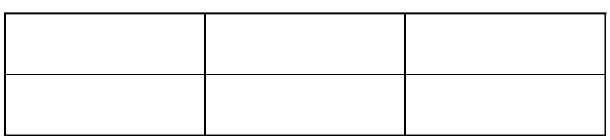
All of these fractions are equivalent because they represent the same value on the number line.

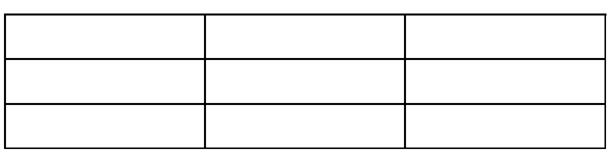


Activity

Shade the rectangles below to find two fractions equivalent to $\frac{1}{3}$.

a) 

b) 

c) 

When we say that $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$, we are saying that these fractions are equivalent; in other words they represent the same rational number.

> If a, b, and c are integers and b and c are not 0, then it is always true that
> $$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$$
> In other words, if the numerator and the denominator of a fraction are **multiplied** by the same nonzero number, the resulting fraction is **equivalent** to the original fraction.

Example 4 : Write $\frac{2}{5}$ as an equivalent fraction with a denominator of 25.

Solution: The denominator of the original fraction is 5. The fraction we are trying to find must have a denominator of 25. We know that if we multiply 5 by 5, we get 25. The property above indicates that we can multiply the denominator by 5 as long as we do the same to the numerator.

$$\frac{2}{5} = \frac{2 \cdot 5}{5 \cdot 5} = \frac{10}{25}$$

Therefore, the fraction $\frac{2}{5}$ is equivalent to $\frac{10}{25}$.

> If a, b, and c are integers and b and c are not 0, then it is always true that
> $$\frac{a}{b} = \frac{a \div c}{b \div c}$$
> In other words, if the numerator and the denominator of a fraction are **divided** by the same nonzero number, the resulting fraction is **equivalent** to the original fraction.

Example 5 : Write $\frac{12}{18}$ as an equivalent fraction with a denominator of 6.

Solution: If we divide the original denominator 18 by 3, we get 6. The property above indicates that if we divide both the numerator and the denominator by 3, we will get an equivalent fraction.

$$\frac{12}{18} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}$$

Therefore, the fraction $\frac{12}{18}$ is equivalent to $\frac{4}{6}$.

Activity 2

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a) Write $\frac{7}{8}$ as an equivalent fraction with a denominator of 48.

b) Write $\frac{24}{40}$ as an equivalent fraction with a denominator of 5.

Let's try and determine when two fractions are equivalent.

Case 1: Fractions With Common Denominators

Consider the fractions $\frac{a}{n}$ and $\frac{b}{n}$. Remember what these fractions represent. If you start with a unit length, $\frac{a}{n}$ represents the rational number obtained by dividing the unit length into n equal segments and then taking a of them and putting them end to end. Putting b of them end to end is the rational number represented by $\frac{b}{n}$. When will the lengths be equal?

This can be stated formally as, **two fractions with the same denominator are equivalent if the numerators are equal.**

Case 2: Fractions With Unlike Denominators

Suppose the denominators are not equal. We observed that we can get a fraction equivalent to $\frac{a}{b}$ by multiplying the top (numerator) and the bottom (denominator) by the same number. In other words, $\frac{a}{b}$ is equivalent to $\frac{4a}{4b}$, $\frac{6a}{6b}$, etc. So, $\frac{a}{b} = \frac{na}{nb}$ for any number n except 0. Consider two fractions, $\frac{a}{b}$ and $\frac{c}{d}$. Then

$\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{cb}{db}$. From case 1, we can state that **if $\mathbf{\frac{a}{b}}$ and $\mathbf{\frac{c}{d}}$ are equivalent , then ad = cb.**

Many people remember this by saying that two fractions are equal when the **cross products** are equal.

Example 6 : Are $\frac{4}{5}$ and $\frac{11}{15}$ equivalent?

Solution: Two fractions are equal if their cross products are equal. Since $4 \cdot 15 \neq 5 \cdot 11$, $\frac{4}{5}$ and $\frac{11}{15}$ are **not** equivalent.

Activity 3

Determine if the following pairs of fractions are equivalent.

a) $\frac{7}{10}, \frac{9}{14}$

b) $\frac{3}{4}, \frac{24}{32}$

Section 3: Comparing Fractions

Now that we can determine if two fractions are equivalent, let's discuss how to determine which of two fractions is greater.

When fractions have the same denominator they are easy to compare. For instance, it is easy to determine that $\frac{7}{12}$ is larger than $\frac{5}{12}$. In both fractions a "whole" is divided into 12 equal parts. In one fraction 7 of these parts are taken and in the other, 5 of the parts are taken, so $\frac{7}{12}$ has to be larger. You can also visualize the two fractions with the figure below to see that $\frac{7}{12}$ has more of the rectangle shaded.

xxxxx	xxxxx	xxxxx			
xxxxx	xxxxx	xxxxx	xxxxx		

xxxxx	xxxxx	xxxxx			
xxxxx	xxxxx				

The key to comparing fractions such as $\frac{3}{7}$ and $\frac{2}{5}$, is to find **equivalent fractions** to each so the two equivalent fractions have the same denominator.

General Rule for Determining Which Fraction is Greater

Case 1: **Fractions With Common Denominators**

$\mathbf{\frac{a}{n} < \frac{b}{n}}$ **if and only if a < b.**

In other words, if the denominators are the same, compare the numerators.

Case 2: **Fractions With Unlike Denominators**

Now consider $\frac{a}{b}$ and $\frac{c}{d}$. We know $\frac{a \cdot d}{b \cdot d} = \frac{c \cdot b}{d \cdot b}$ so we can

then state that $\mathbf{\frac{a}{b} < \frac{c}{d}}$ **if and only if a · d < c · b.**

In other words, if the denominators are different, compare the cross products.

Example 7 : Which fraction is greater: $\frac{3}{8}$ or $\frac{7}{17}$?

Solution: Let's start by finding equivalent fractions for both that have the same denominator. Multiply the top and the bottom of $\frac{3}{8}$ by 17 and multiply the top and the bottom of $\frac{7}{17}$ by 8. This gives us

$$\frac{3}{8} = \frac{3 \cdot 17}{8 \cdot 17} = \frac{51}{136} \text{ and } \frac{7}{17} = \frac{7 \cdot 8}{17 \cdot 8} = \frac{56}{136}.$$

Since 56 is greater than 51, you can see that $\frac{7}{17}$ is greater than $\frac{3}{8}$.

The shortcut method is given in case 2 above.

$$\frac{3}{8} < \frac{7}{17} \text{ because } 3 \cdot 17 < 7 \cdot 8.$$

Example 8 : Which fraction is greater: $\frac{2}{15}$ or $\frac{7}{53}$?

Solution: Since $2(53) > 7(15)$, $\frac{2}{15}$ is greater than $\frac{7}{53}$.

Activity 4

State which fraction is greater.

a) $\frac{2}{5}$ or $\frac{4}{7}$

b) $\frac{13}{20}$ or $\frac{9}{15}$

Section 4: Simplifying Fractions to Lowest Terms

By looking at the following examples,

$\frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}$

$\frac{6}{7} = \frac{6 \cdot 9}{7 \cdot 9} = \frac{54}{63}$ can you think of a way to start with $\frac{54}{63}$ and simplify to get $\frac{6}{7}$?

To simplify a fraction to lowest terms, we have to divide the numerator and the denominator by all the **factors** they have in common. Recall that factors are numbers that are multiplied together. When we divide the numerator and the denominator of a fraction by the same nonzero number, we get an equivalent fraction. So to simplify a fraction all we have to do is recognize what factors the numerator and the denominator have in common and then divide the numerator and the denominator by the common factors.

Definition: A fraction is said to be in **lowest terms** if the numerator and the denominator have no factors in common other than the number 1.

Example 9 : Write $\frac{12}{15}$ in the lowest terms.

Solution: We need to find the factors of 12 and 15 so that we can find one they have in common.

Factors of 12 are: 1, 2, 3, 4, 6, 12

Factors of 15 are: 1, 3, 5, 15

Since 3 is the only factor they have in common, we will divide the numerator and denominator by 3.

$\frac{12 \div 3}{15 \div 3} = \frac{4}{5}$ *In lowest terms, since 4 and 5 have no other factor besides 1 in common.*

Example 10 : Write $\frac{16}{24}$ in lowest terms.

Solution: Factors of 16 are: 1, 2, 4, 8, 16

Factors of 24 are: 1, 2, 3, 4, 6, 8, 12, 24

Since 16 and 24 have 2, 4, and 8 in common, we will choose the highest factor, 8.

$\frac{16 \div 8}{24 \div 8} = \frac{2}{3}$. *In lowest terms, since 2 and 3 have no other factor besides 1 in common.*

Notice that you could also write $\frac{16}{24}$ *in lowest terms, by simply dividing numerator and denominator by 2 until they have no factor in common but 1.*

$$\frac{16 \div 2}{24 \div 2} = \frac{8 \div 2}{12 \div 2} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3} .$$

Activity 5

Simplify the following fractions. Show all your work.

a) $\frac{54}{72}$ b) $\frac{16}{28}$

Section 5: Adding and Subtracting Fractions

To Add or Subtract Fractions with Like Denominators:

(1) Add or subtract the numerator and place the answer over the common denominator.

(2) Write the answer in lowest terms.

Example 11 : a) Add: $\frac{3}{8}+\frac{1}{8}$ b) Subtract: $\frac{7}{10}-\frac{3}{10}$

Solution: a) Since the denominators are the same, add the numerators and keep the denominator the same.

$$\frac{3}{8}+\frac{1}{8} = \frac{4}{8}$$

Since $\frac{4}{8}$ is not in lowest terms, divide numerator and denominator by 4 to get $\frac{1}{2}$.

b) Again the denominators are the same, so we just subtract the numerators and place that answer over the common denominator.

$$\frac{7}{10}-\frac{3}{10} = \frac{4}{10}$$

Since $\frac{4}{10}$ is not in lowest terms, divide numerator and denominator by 2 to get $\frac{2}{5}$.

To Add or Subtract Fractions with Unlike Denominators:

(1) Find equivalent fractions for each original fraction, so that both fractions have a common denominator.

(2) Follow the procedure for like denominators.

Example 12 : Add: $\frac{5}{12}+\frac{1}{18}$

Solution: Since we have unlike denominators, we need to determine a common denominator between 12 and 18. To do this, we need to look at the multiples of 12 and 18 and find the smallest common multiple.

Multiples of 12 are: 12, 24, 36, 48, …

Multiples of 18 are: 18, 36, 54, 72, …

Since 36 is the 1st multiple in common, that will be our common denominator.

You could also just multiply their denominators together to find a common multiple but this often leads to large numbers which are not easily simplified.

Next , we need to write equivalent fractions for each of our original fractions, having a common denominator of 36.

$$\frac{5\cdot 3}{12\cdot 3}=\frac{15}{36}$$ *Since 12 · 3 = 36, we multiply numerator and denominator by 3.*

$$\frac{1\cdot 2}{18\cdot 2}=\frac{2}{36}$$ *Since 18 · 2 = 36, we multiply numerator and denominator by 2.*

Now that we have common denominators, we can add the numerators and place that over the common denominator,

$$\frac{15+2}{36}=\boxed{\frac{17}{36}}$$

Example 13 : Subtract: $\frac{4}{15}-\frac{2}{9}$

Solution: Multiples of 15 are: 15, 30, 45, 60,…

Multiples of 9 are: 9, 18, 27, 36, 45,…

Since 45 is the 1st multiple in common, that will be our common denominator.

$$\frac{4\cdot 3}{15\cdot 3}=\frac{12}{45}$$ *Multiply numerator and denominator by 3.*

$$-\ \frac{2\cdot 5}{9\cdot 5}=\frac{10}{45}$$ *Multiply numerator and denominator by 5.*

$$\boxed{\frac{2}{45}}$$ *Subtract numerators and place answer over common denominator.*

Activity 6

a) Add: $\frac{3}{8}+\frac{2}{5}$

b) Subtract: $\frac{7}{15}-\frac{3}{10}$

c) Add: $\frac{7}{9}+\frac{1}{6}$

d) Subtract: $\frac{2}{3}-\frac{3}{8}$

Section 6: Multiplying Fractions

To Multiply Fractions:

(1) Divide out any common factors between any numerator and any denominator.

(2) Multiply remaining numbers in the numerator, then multiply remaining numbers in the denominator.

(3) Place the product of the numerators over the product of the denominators.

(4) Check that the answer is in lowest terms.

Example 14 : Multiply: $\frac{3}{5} \cdot \frac{2}{7}$

Solution: Since there are no common factors between any of the numerators and denominators, we simply multiply the numerators, then multiply denominators.

$$\frac{3}{5} \cdot \frac{2}{7} = \frac{3 \cdot 2}{5 \cdot 7} = \boxed{\frac{6}{35}}$$ which is in lowest terms.

Example 15 : Multiply: $\frac{25}{32} \cdot \frac{8}{20}$

Solution:

$$\frac{{}^{5}2\not{5}}{32} \cdot \frac{8}{2\not{0}_{4}}$$ *Divide 20 and 25 by their common factor of 5.*

$$\frac{{}^{5}2\not{5}}{3\not{2}_{4}} \cdot \frac{\not{8}^{1}}{2\not{0}_{4}}$$ *Divide 8 and 32 by their common factor of 8.*

$$\frac{5 \cdot 1}{4 \cdot 4} = \boxed{\frac{5}{16}}$$ *Multiply remaining numbers in the numerator and denominator.*

Answer is in lowest terms.

Activity 7

Multiply:

a) $\frac{7}{8} \cdot \frac{24}{35}$

b) $\frac{9}{20} \cdot \frac{8}{15}$

c) $\frac{5}{18} \cdot \frac{8}{35}$

d) $\frac{12}{25} \cdot \frac{15}{16}$

Section 7: Dividing Fractions

Definition: If $\frac{a}{b}$ is a fraction, the fraction $\frac{b}{a}$ is called the reciprocal of $\frac{a}{b}$.

To Divide Fractions:

(1) Rewrite the 1st fraction as it is given.

(2) Change the division sign to a multiplication sign.

(3) Write the reciprocal of the 2nd fraction.

(4) Use the rules for multiplying fractions.

Example 16 : Divide: $\frac{1}{5} \div \frac{2}{3}$

Solution: $\frac{1}{5} \cdot \frac{3}{2}$ *Multiply by the reciprocal.*

$\frac{1 \cdot 3}{5 \cdot 2} = \boxed{\frac{3}{10}}$ *Multiply numerators, then multiply denominators.*

Answer in lowest terms.

Example 17 : Divide: $\frac{13}{28} \div \frac{26}{35}$

Solution: $\frac{13}{28} \cdot \frac{35}{26}$ *Multiply by the reciprocal.*

$\frac{{}^{1}\cancel{13}}{\cancel{28}_{4}} \cdot \frac{\cancel{35}^{5}}{\cancel{26}_{2}}$ *Divide out any common factors between any numerator and any denominator (divide 28 and 35 by 7 ; 13 and 26 by 13).*

$\frac{1 \cdot 5}{4 \cdot 2} = \boxed{\frac{5}{8}}$ *Multiply numerators, then multiply denominators.*

Answer is in lowest terms.

Example 18 : Divide: $\frac{3}{2} \div 9$

Solution: $\frac{3}{2} \div \frac{9}{1}$ *Write 9 as a fraction by placing a 1 in the denominator.*

$\frac{3}{2} \cdot \frac{1}{9}$ *Multiply by the reciprocal.*

$\frac{{}^{1}\cancel{3}}{2} \cdot \frac{1}{\cancel{9}_{3}}$ *Divide 3 and 9 by 3.*

$\frac{1 \cdot 1}{2 \cdot 3} = \boxed{\frac{1}{6}}$ *Multiply numerators, then multiply denominators.*

Answer is in lowest terms.

Activity 8

Divide:

a) $\frac{3}{8} \div \frac{15}{4}$

b) $\frac{24}{25} \div \frac{6}{5}$

c) $\frac{4}{9} \div 6$

Section 8: Mixed Numbers

A number of the form $3\frac{2}{5}$ is called a mixed number because it is composed of a whole number and a fraction. Sometimes people misinterpret it to mean 3 times $\frac{2}{5}$. In general, a number of the form $A\frac{b}{c}$ means $A+\frac{b}{c}$, so $3\frac{2}{5}=3+\frac{2}{5}$. Mixed numbers are helpful to get a better idea of the size of a number. For example, when you look at an improper fraction such as $\frac{341}{17}$, it is difficult to get a feeling for its size. However, if we write $\frac{341}{17}$ as the mixed number, $21\frac{4}{17}$, we have a better feeling for its size.

To Change from an Improper Fraction to a Mixed Number:

(1) Divide the denominator into the numerator. This quotient becomes the whole number part of the mixed number.

(2) If the denominator does not divide evenly into the numerator, place the remainder over the denominator. This becomes the fractional part of the mixed number.

Example 19 : Change $\frac{37}{5}$ to a mixed number.

Solution:

$$\begin{array}{r} 7 \\ 5\overline{)37} \\ \underline{35} \\ 2 \end{array}$$

Divide the denominator into the numerator.

Since 2 is the remainder, place that over the denominator 5. Write the mixed number.

Therefore, $\frac{37}{5} = \boxed{7\frac{2}{5}}$.

To Change from a Mixed Number to an Improper Fraction:

(1) Multiply the whole number and the denominator, add the result to the numerator.

(2) Place the answer from step (1) over the denominator.

Example 20 : Change $5\frac{4}{9}$ to an improper fraction.

Solution: $5\frac{4}{9}=\frac{5\bullet 9+4}{9}=\frac{45+4}{9}=\frac{49}{9}$

Therefore, $5\frac{4}{9}=\boxed{\frac{49}{9}}$.

Activity 9

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a) Change the following improper fractions to mixed numbers.

i) $\frac{41}{9}$ ii) $\frac{527}{21}$

b) Change the following mixed numbers to improper fractions.

i) $8\frac{3}{7}$ ii) $12\frac{5}{9}$

To Add and Subtract Mixed Numbers:

The easiest method is to change the mixed numbers to improper fractions and use the rules given for adding and subtracting fractions.

Example 21 : Add: $3\frac{2}{5}+2\frac{2}{3}$

Solution: $3\frac{2}{5}+2\frac{2}{3}=\frac{17}{5}+\frac{8}{3}=\frac{51}{15}+\frac{40}{15}=\frac{91}{15}=6\frac{1}{15}$

Therefore, $3\frac{2}{5}+2\frac{2}{3}=\boxed{6\frac{1}{15}}$.

Example 22 : Subtract: $4\frac{1}{6}-2\frac{1}{4}$

Solution: $4\frac{1}{6}-2\frac{1}{4} = \frac{25}{6}-\frac{9}{4}=\frac{50}{12}-\frac{27}{12}=\frac{23}{12}=1\frac{11}{12}$

Therefore, $4\frac{1}{6}-2\frac{1}{4} = \frac{23}{12} = \boxed{1\frac{11}{12}}$.

Activity 10

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a) Add: $4\frac{1}{5}+3\frac{1}{2}$

b) Subtract: $12\frac{3}{5}-5\frac{1}{2}$

c) Add: $7\frac{5}{8}+4\frac{1}{6}$

d) Subtract: $7\frac{1}{3}-2\frac{3}{5}$

To Multiply and Divide Mixed Numbers:

Change all mixed numbers to improper fractions and use the rules given for multiplying and dividing fractions.

Example 23 : Multiply: $2\frac{2}{3}\bullet 1\frac{1}{2}$

Solution: $2\frac{2}{3}\bullet 1\frac{1}{2}=\frac{8}{3}\bullet\frac{3}{2}=\frac{{}^{4}\cancel{8}}{{}_{1}\cancel{3}}\bullet\frac{\cancel{3}^{1}}{\cancel{2}_{1}}=\frac{4}{1}=4$

Therefore, $2\frac{2}{3}\bullet 1\frac{1}{2} = \boxed{4}$.

Example 24 : Divide: $6\frac{1}{3} \div 4\frac{2}{9}$

Solution: $6\frac{1}{3} \div 4\frac{2}{9} = \frac{19}{3} \div \frac{38}{9} = \frac{19}{3} \bullet \frac{9}{38} = \frac{\cancel{19}^{1}}{\cancel{3}_{1}} \bullet \frac{\cancel{9}^{3}}{{}^{2}\cancel{38}} = \frac{3}{2} = 1\frac{1}{2}$

Therefore, $6\frac{1}{3} \div 4\frac{2}{9} = \boxed{1\frac{1}{2}}$.

Activity 11

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a) Multiply: $2\frac{3}{4} \bullet 5\frac{3}{8}$

b) Divide: $\frac{3}{5} \div 2\frac{3}{4}$

c) Multiply: $6\frac{3}{7} \bullet 2\frac{1}{3}$

d) Divide: $7\frac{1}{9} \div 4\frac{2}{3}$

Section 9: Fraction Problems

In this section, we will solve word problems involving fractions.

Example 25 : Three-fifths of the students at Keene State College are female. If there are 4015 students at Keene State College, how many students are male?

Solution: The key to solving this problem is to notice the words "Three-fifths of the students." When you are taking a fraction of a quantity, you need to multiply.

$$\frac{3}{5} \text{ of the students} = \frac{3}{5} \bullet 4015 \qquad \textit{replace students with 4015}$$

$$= \frac{3}{\cancel{5}_1} \bullet \frac{\cancel{4015}^{803}}{1} \qquad \textit{divide 5 and 4015 by their common factor of 5}$$

$$= 3 \bullet 803 \qquad \textit{multiply 3 and 803}$$

$$= 2409$$

Therefore, there are 2409 female students at KSC. To determine how many males there are, subtract 2409 from 4015 to get 1606 males at KSC.

Example 26 : You have 14 ft of wood. To make a frame, you need $3\frac{1}{4}$ feet of wood. How many frames can you make with the 14 ft of wood?

Solution: Sometimes it helps to draw a picture when trying to solve a word problem. In this case, we can draw a straight line to represent the 14 feet of wood and think about what it would mean to cut $3\frac{1}{4}$ feet pieces out of the wood.

Mathematically, we could subtract $3\frac{1}{4}$ from 14 and then from the remaining pieces until we have a piece that is less than $3\frac{1}{4}$. This process would take a few steps, so instead we are going to use the shortcut for repeated subtraction, which is division.

$$14 \div 3\frac{1}{4} = \frac{14}{1} \div \frac{13}{4}$$ *change the mixed number to an improper fraction*

$$= \frac{14}{1} \cdot \frac{4}{13}$$ *multiply by the reciprocal*

$$= \frac{56}{13} = 4\frac{4}{13}$$ *multiply numerators and then multiply denominators and write answer as a mixed number*

Therefore, you would be able to make 4 frames out of the 14 feet of wood. The $\frac{4}{13}$ is the fraction of a fifth frame that could be made from the leftover wood.

Example 27 : On Monday, Austin practiced his trumpet for $\frac{1}{4}$ of an hour. On Tuesday, he practiced his trumpet for $\frac{5}{6}$ of an hour and on Wednesday, he practiced for $\frac{1}{2}$ of an hour. If Austin is supposed to practice for 2 hours a week, what fraction of an hour must he practice on Thursday before his Friday lesson?

Solution: To solve this problem, we must first add all the hours he has practiced and then subtract from 2.

$$\frac{1}{4} + \frac{5}{6} + \frac{1}{2} = \frac{3}{12} + \frac{10}{12} + \frac{6}{12}$$ *find a common denominator and write equivalent fractions*

$$= \frac{19}{12}$$ *add numerators and write over common denominator*

Now we need to subtract $\frac{19}{12}$ from 2 to determine what fraction of an hour he must still practice.

$$2 - \frac{19}{12} = \frac{24}{12} - \frac{19}{12} = \frac{5}{12}$$

Therefore, Austin must practice $\frac{5}{12}$ of an hour on Thursday.

a) The contents of a box of canned soup weigh 35 pounds. If each can weighs $1\frac{2}{5}$ pounds, find how many cans of soup are in the box.

b) The KSC track team ran a relay race that was a total of 5 meters. If the first 3 team members ran lengths of $1\frac{1}{5}$, $1\frac{8}{25}$, and $1\frac{2}{5}$ meters, how many meters was left for the anchor to run?

c) Your rent is $\frac{2}{7}$ of your monthly salary. If your annual salary is $29,568, how much is your rent each month?

Section 10: Ratios

In any field, size comparisons are often necessary. A useful method of comparing things is by a **ratio**, which is a concise way to express the relative sizes of two measures. For example, you know that there are 12 inches in one foot. The ratio is 12 inches to 1 foot, which can also be written as 12:1 or as $\frac{12 \text{ inches}}{1 \text{ foot}}$.

In geometry, the ratio of the circumference *(distance around a circle)* of a circle to the diameter is π (pi). The probability of an event occurring is a ratio. Probabilities are part-to-whole ratios: the number of favorable outcomes to the total number of possible outcomes. The probability of throwing a die and getting a 5 is 1 to 6. One favorable outcome to a total of six possible outcomes.

In physics, the laws governing the forces of pulleys, levers, and gears all involve ratios. **Boyle's law** says that the volume of a fixed amount of gas is determined by the ratio of the temperature to the pressure.

Even in music, notes are in the ratio of 4 notes to a measure, 12 tones to an octave. Pleasant-sounding chords involve tones with wavelengths in special ratios.

When a ratio compares the measures of two different things it is called a **rate**. In a rate the units are different. When you have a ratio of money to the measure of a quantity you have a rate. For example, 12 ounces for $2.35 is a rate. Another common example is 55 miles per hour or 32 miles per gallon.

Example 28 : A driven gear has 18 teeth and the driving gear has 8 teeth. What is the gear ratio of the driven gear to the driving gear?

Solution: $\frac{\text{driven gear}}{\text{driving gear}} = \frac{18 \text{ teeth}}{8 \text{ teeth}} = \frac{9}{4} = \frac{2\frac{1}{4}}{1} = 2\frac{1}{4}$

Note: The above example is important because some ratios are always expressed as a quantity compared to 1.

There are three common ways ratios are written:

$\frac{a}{b}$, a to b, and a : b.

Example 29 : In 2003 there were 2,500 females and 1,812 males at Keene State College. What was the ratio of females to males?

Solution: $\frac{2500 \ \textit{females}}{1812 \ \textit{males}}$ could also be written as $\frac{1.38 \ \textit{females}}{1 \ \textit{male}}$**,** or 1.38, which tells us that there are about one-and-one-half times as many females at Keene State than males.

Activity 13

Suppose in the last activity you took the ratio of males to females, interpret this ratio.

Example 30 : A 14 ounce box of cereal costs \$3.49 and a 10 ounce box of the same cereal costs \$2.59. Which is the better buy?

Solution: To determine which cereal is the better buy, we need to calculate the cost of one ounce for each box of cereal. To do that, we simply divide the cost of the cereal by the number of ounces in the box.

$$\frac{\$3.49}{14\ oz} = \$\,0.249\,/\,ounce \qquad \frac{\$2.59}{10\ oz} = \$\,0.259\,/\,ounce$$

Since \$0.249 is less than \$0.259, then the 14 ounce box of cereal is the better buy.

Activity 14

An 18 ounce bottle of hand lotion costs \$3.69 and a 12 ounce bottle of the same hand lotion costs \$1.79. Which is the better buy?

Ratios are important when discussing changes in a quantity over time. For example, the number of students at Keene State in 1985 was 3,267 and by 1998 it had grown to 4,321. The net gain in students was 1,054. Instead of computing the difference we could take the ratio of the student population in 1998 to the student population in 1985. This ratio, 1.3, tells us the population grew by a factor of 1.3.

Another way to describe the change is by calculating the **percent change**.

$$\textbf{Percent change} = 100\left(\frac{\textit{New Size - Original Size}}{\textit{Original Size}}\right)$$

The percent change for Keene State from 1985 to 1998 was

$$100\left(\frac{4321 - 3267}{3267}\right) = 32.3\%.$$

Activity 15

Suppose you invest \$1200 in the stock market. At the end of the year you sell the stock and receive \$1500 after paying your commission. What is percent gain?

Exercises for Fractions and Ratios

1. a) Write $\frac{5}{3}$ as a fraction with a denominator of 9.

   b) Write $\frac{8}{32}$ as a fraction with a denominator of 16.

2. Simplify the following fractions.

   a) $\frac{4}{14}$ b) $\frac{14}{35}$ c) $\frac{24}{48}$ d) $\frac{144}{196}$ e) $\frac{82}{104}$ f) $\frac{325}{725}$

3. During fall 2001, 1200 students visited the Math Center for tutoring, testing or for a PCA study session. During fall 2002, 1600 students visited the Math Center for the same services. What was the percent change from fall 2001 to fall 2002?

4. Of 45 vehicles in the parking lot, 27 are cars and 18 are trucks. What is the ratio of trucks to vehicles?

5. Give four fractions equivalent to $\frac{4}{9}$.

6. Circle the larger of the two fractions.

   a) $\frac{4}{7}$ $\frac{6}{11}$ b) $\frac{9}{13}$ $\frac{5}{7}$

7. Find the sum of $\frac{3}{7}$ and $\frac{1}{9}$.

8. Subtract $\frac{1}{9}$ from $\frac{3}{7}$.

9. Compute the following.

   a) $\frac{7}{12}+\frac{4}{9}$ b) $\frac{3}{14}+\frac{6}{7}$ c) $\frac{4}{9}-\frac{2}{15}$ d) $\frac{7}{10}-\frac{2}{45}$

10. Simplify the following fractions.

    a) $\frac{75}{90}$ b) $\frac{84}{132}$

11. Add the following

a) $\frac{4}{9} + \frac{2}{3}$ b) $\frac{1}{3} + \frac{3}{5}$ c) $\frac{4}{5} + \frac{4}{7}$ d) $\frac{3}{16} + \frac{1}{24}$

12. Multiply the following.

a) $\frac{4}{9} \times \frac{2}{3}$ b) $\frac{1}{3} \times \frac{3}{5}$ c) $\frac{4}{5} \times \frac{4}{7}$ d) $\frac{21}{34} \times \frac{8}{9}$

13. Compute the following.

a) $3\frac{2}{5} + 5\frac{3}{8}$ b) $5\frac{3}{8} - 3\frac{2}{5}$ c) $5\frac{3}{8} \bullet 2\frac{2}{7}$

d) $3\frac{2}{5} \div 1\frac{9}{25}$ e) $3\frac{3}{4} \div 2\frac{2}{5}$

14. Compute the following.

a) $\frac{3}{2} \div \frac{1}{5}$ b) $\frac{3}{7} \div \frac{2}{5}$ c) $\frac{5}{3} \div \frac{1}{2}$

15. Three-fourths of the students at Keene State live in the dorms, $\frac{2}{3}$ of the rest walk to school. What fraction of the students do not live in a dorm or do not walk to school?

16. What is the ratio of the height of a tree 60 ft tall and the length of its shadow, 86 ft long?

17. A pair of basketball shorts requires $\frac{3}{4}$ yards of material. How many pairs of shorts can be made from 15 yards of material?

18. An estate was left to four children. One received $\frac{1}{8}$ of the estate, the second $\frac{1}{6}$, and the third $\frac{5}{12}$. How much did the fourth child receive?

19. Which is the better buy?

A package of 25 blank CD's for $5.50 or a package of 50 CD's for $10.50.

FRACTIONS and RATIOS

Activity 1:

a)

XXXXXXXXX XXXXXXXXX XXXXXXXXX		

b)

XXXXXXXXX XXXXXXXXX		
XXXXXXXXX XXXXXXXXX		

c)

XXXXXXXXX		
XXXXXXXXX		
XXXXXXXXX		

a) $\frac{1}{3}$ b) $\frac{2}{6}$ c) $\frac{3}{9}$

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Activity 2:

a) $\frac{7}{8} = \frac{42}{48}$ b) $\frac{24}{40} = \frac{3}{5}$

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Activity 3:

a) Since 14 • 7 = 98 and 9 • 10 = 90, then $\frac{7}{10}$ and $\frac{9}{14}$ are not equivalent.

b) Since 3 • 32 = 96 and 24 • 4 = 96, then $\frac{3}{4}$ and $\frac{24}{32}$ are equivalent .

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Activity 4:

a) $\frac{2}{5} < \frac{4}{7}$ because 2 • 7 < 5 • 4 b) $\frac{13}{20} > \frac{9}{15}$ because 13 • 15 > 9 • 20

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Activity 5:

a) $\frac{54}{72} \div \frac{9}{9} = \frac{6}{8} \div \frac{2}{2} = \frac{3}{4}$ b) $\frac{16}{28} \div \frac{4}{4} = \frac{4}{7}$

Activity 6:

a)

$$\frac{3}{8}=\frac{15}{40}$$
$$+\quad\frac{2}{5}=\frac{16}{40}$$
$$\frac{31}{40}$$

b)

$$\frac{7}{15}=\frac{28}{60}$$
$$-\quad\frac{3}{10}=\frac{18}{60}$$
$$\frac{10}{60}=\frac{1}{6}$$

c)

$$\frac{7}{9}=\frac{14}{18}$$
$$+\quad\frac{1}{6}=\frac{3}{18}$$
$$\frac{17}{18}$$

d)

$$\frac{2}{3}=\frac{16}{24}$$
$$-\quad\frac{3}{8}=\frac{9}{24}$$
$$\frac{7}{24}$$

****

Activity 7:

a) $\frac{{}^{1}\cancel{7}}{{}_{1}\cancel{8}}\bullet\frac{\cancel{24}^{3}}{\cancel{35}_{5}}=\frac{3}{5}$

b) $\frac{{}^{3}\cancel{9}}{{}_{5}\cancel{20}}\bullet\frac{\cancel{8}^{2}}{\cancel{15}_{5}}=\frac{6}{25}$

c) $\frac{{}^{1}\cancel{5}}{{}_{9}\cancel{18}}\bullet\frac{\cancel{8}^{4}}{\cancel{35}_{7}}=\frac{4}{63}$

d) $\frac{{}^{3}\cancel{12}}{{}_{5}\cancel{25}}\bullet\frac{\cancel{15}^{3}}{\cancel{16}_{4}}=\frac{9}{20}$

****

Activity 8:

a) $\frac{3}{8}\div\frac{15}{4}=\frac{{}^{1}\cancel{3}}{{}_{2}\cancel{8}}\bullet\frac{\cancel{4}^{1}}{\cancel{15}_{5}}=\frac{1}{10}$

b) $\frac{24}{25}\div\frac{6}{5}=\frac{{}^{4}\cancel{24}}{{}_{5}\cancel{25}}\bullet\frac{\cancel{5}^{1}}{\cancel{6}_{1}}=\frac{4}{5}$

c) $\frac{4}{9}\div 6=\frac{{}^{2}\cancel{4}}{9}\bullet\frac{1}{\cancel{6}_{3}}=\frac{2}{27}$

Activity 9:

a) i) $\frac{41}{9} = 4\frac{5}{9}$ ii) $\frac{527}{21} = 25\frac{2}{21}$

b) i) $8\frac{3}{7} = \frac{59}{7}$ ii) $12\frac{5}{9} = \frac{113}{9}$

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Activity 10:

a) $4\frac{1}{5} + 3\frac{1}{2} = \frac{21}{5} + \frac{7}{2} = \frac{42}{10} + \frac{35}{10} = \frac{77}{10} = 7\frac{7}{10}$

b) $12\frac{3}{5} - 5\frac{1}{2} = \frac{63}{5} - \frac{11}{2} = \frac{126}{10} - \frac{55}{10} = \frac{71}{10} = 7\frac{1}{10}$

c) $7\frac{5}{8} + 4\frac{1}{6} = \frac{61}{8} + \frac{25}{6} = \frac{183}{24} + \frac{100}{24} = \frac{283}{24} = 11\frac{19}{24}$

d) $7\frac{1}{3} - 2\frac{3}{5} = \frac{22}{3} - \frac{13}{5} = \frac{110}{15} - \frac{39}{15} = \frac{71}{15} = 4\frac{11}{15}$

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Activity 11:

a) $2\frac{3}{4} \bullet 5\frac{3}{8} = \frac{11}{4} \bullet \frac{43}{8} = \frac{473}{32} = 14\frac{25}{32}$ b) $\frac{3}{5} \div 2\frac{3}{4} = \frac{3}{5} \div \frac{11}{4} = \frac{3}{5} \bullet \frac{4}{11} = \frac{12}{55}$

c) $6\frac{3}{7} \bullet 2\frac{1}{3} = \frac{{}^{15}\cancel{45}}{{}_{1}\cancel{7}} \bullet \frac{\cancel{7}^{1}}{\cancel{3}_{1}} = 15$ d) $7\frac{1}{9} \div 4\frac{2}{3} = \frac{64}{9} \div \frac{14}{3} = \frac{{}^{32}\cancel{64}}{{}_{3}\cancel{9}} \bullet \frac{\cancel{3}^{1}}{\cancel{14}_{7}} = \frac{32}{21} = 1\frac{11}{21}$

****

Activity 12:

a) To determine how many cans of soup are in the box, divide the total weight of the box by the weight of each can.

$$35 \div 1\frac{2}{5} = 35 \div \frac{7}{5} = \frac{{}^{5}\cancel{35}}{1} \bullet \frac{5}{\cancel{7}_{1}} = 25$$

Therefore, there are 25 cans of soup in the box.

b) To determine how many meters the anchor has to run, add up how many meters the first 3 team members already ran.

$$1\frac{1}{5}+1\frac{8}{25}+1\frac{2}{5}=\frac{6}{5}+\frac{33}{25}+\frac{7}{5}=\frac{30}{25}+\frac{33}{25}+\frac{35}{25}=\frac{98}{25}$$

Next, subtract $\frac{98}{25}$ meters from 5 meters, the length of the race.

$$5-\frac{98}{25}=\frac{125}{25}-\frac{98}{25}=\frac{27}{25}=1\frac{2}{25}$$

Therefore, the anchor must run $1\frac{2}{25}$ meters.

c) To determine how much your rent is each month, you first need to determine how much you earn each month by dividing your annual salary by 12.

$$\frac{29568}{12}=2464$$ So, your monthly salary is $2464.

Next, you need to recognize the key words, "$\frac{2}{7}$ of your monthly salary" and multiply $\frac{2}{7}$ and 2464.

$$\frac{2}{\cancel{7}_1}\bullet\frac{\cancel{2464}^{352}}{1}=2\bullet 352=704$$

Therefore, your rent is $704 per month.

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Activity 13:

$\frac{1812}{2500}=\frac{.7}{1}$, which tells us that there are $\frac{7}{10}$ as many males as females at Keene State.

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Activity 14:

$\frac{\$3.69}{18\ oz}=\$0.205/ounce$ $\frac{\$1.79}{12\ oz}=\$0.149/ounce$

Since $0.149 is less than $0.205, then the 12 ounce bottle is the better buy.

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Activity 15:

Percent change = 100 $\left(\frac{1500-1200}{1200}\right)=100\,(25)=25\%$ gain.