

Math 171 Proficiency Packet on *Integers & Exponents*

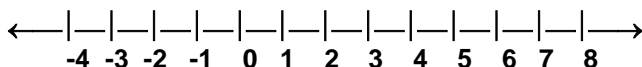
Section 1: Integers

For many of man's purposes the set of whole numbers $W = \{0, 1, 2, \dots\}$ is inadequate. It become necessary to invent negative numbers and extend the set of whole numbers to the set of integers:

$$I = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

Integers are used to record debits and credits, profits and losses, changes in the stock market, degrees above zero and degrees below zero in measuring temperature, yards lost and yards gained in a football game, and so on.

A number line can be used to represent the integers. Start out by choosing an arbitrary point on the number line to represent 0. Once we know unit distance we can label successive points on the right side of 0 with **positive integers** (the natural or counting numbers) and points on the left with successive **negative integers** (-1, -2, -3, ...).



This number line model of the integers is useful in showing their relative sizes (**smaller integers lie to the left of larger integers**) and for illustrating that the nonnegative integers can be identified with the whole numbers. Note that if **a** and **b** are integers, and if **a** is larger than **b**, **a** is to the right of **b** on the number line. This is written symbolically as **b < a** or **a > b**.

Example 1: Order the following integers from the smallest to largest. -1, -13, 5, 0, 8, -12

Solution: $-13 < -12 < -1 < 0 < 5 < 8$

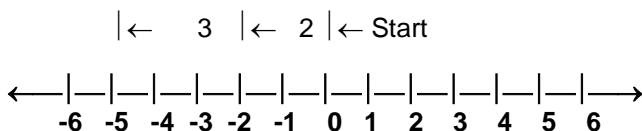
Now You Try (Section 1.1)

Order the following integers from smallest to largest. It is helpful to visualize where these integers belong on a number line.

-5, 2, -7, 0, 6, -1

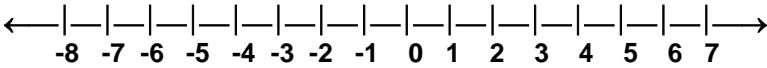
(Answers to **Now You Try** (Section 1.1) are found on page 16.)

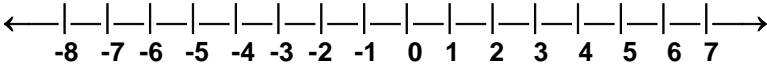
Now let's look at **addition and subtraction** of the integers using the number line. To find $-3 + -2$ using a number line, picture this sum as the result of starting at the origin and moving two units to the left and then another 3 units again to the left. Doing this we end up at -5. So, $-3 + -2 = -5$.

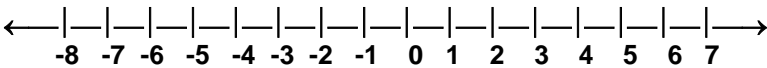


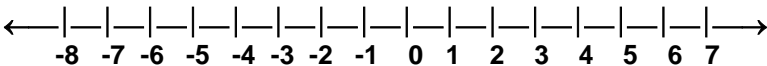
Now You Try (Section 1.2)

Calculate the following using a number line.

1) $-2 + -5$ 

2) $-3 + -4$ 

3) $-4 + 7$ 

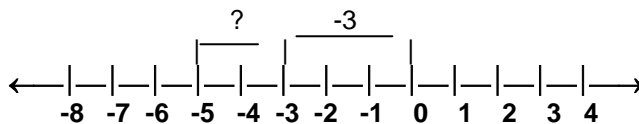
4) $5 + -4$ 

(Answers to **Now You Try** (Section 1.2) are found on page 16.)

One way to look at subtraction is with a **missing addend interpretation**. When we do the subtraction $5 - 2$ we are really looking for the number we would add to 2 to get 5. Therefore, $5 - 2 = 3$ since $3 + 2 = 5$.

Example 1: Compute $-5 - -3$.

Solution: Using the **missing addend interpretation**, we must find the solution to $\square + -3 = -5$. A number line may be useful in visualizing this equation

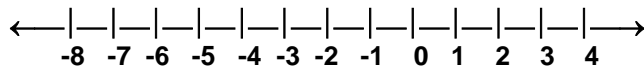


We are at -3 and are being asked, "what has to be added to -3 so that we end up at -5?" Since we have to move to the left 2 units, $-2 + -3 = -5$. Therefore, $-5 - -3 = -2$.

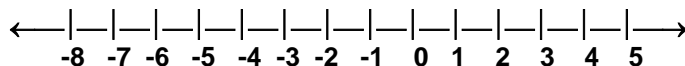
Now You Try (Section 1.3)

Calculate the following. Use a number line to check your solutions.

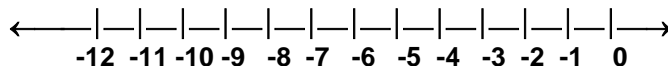
1) $-6 - -5 = \underline{\hspace{2cm}}$ since $+ -5 = -6$.



2) $-3 - -8 = \underline{\hspace{2cm}}$ since $+ -8 = -3$.



3) $-6 - 6 = \underline{\hspace{2cm}}$ since $+ 6 = -6$.



(Answers to **Now You Try** (Section 1.3) are found on page 17.)

Section 2: Addition of Integers

Before we discuss addition of integers, we need to define the absolute value of a number.

Absolute Value: The distance from zero to a number on the number line. The absolute value will either be positive or zero. The symbol for absolute value is two vertical bars $| |$.

Example 1: Evaluate:

a) $|-5|$

b) $|8|$

c) $|0|$

d) $|7 + 3|$

- Solution:**
- a) Since -5 is 5 units away from zero the absolute value of -5 equals 5. ($|-5| = 5$)
 - b) Since 8 is 8 units away from zero, the absolute value of 8 equals 8. ($|8| = 8$)
 - c) $|0| = 0$
 - d) $|7 + 3| = |10| = 10$

Now You Try (Section 2.1)

Evaluate:

- 1) $|-7|$
- 2) $|5 - 5|$
- 3) $|24|$

(Answers to **Now You Try** (Section 2.1) are found on page 17.)

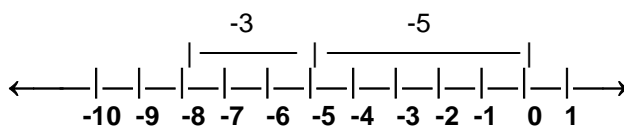
Now that you understand the absolute value, we can discuss the rules for adding integers.

Adding Two Integers

Case 1: **Same Signs** → **ADD** the absolute value of the integers. Use the common sign in the answer.

- Example 1:**
- a) $7 + 9 = 16$
 - b) $-5 + -3 = |-5| + |-3| = 5 + 3 = 8$ since they are both negative, the answer is -8.

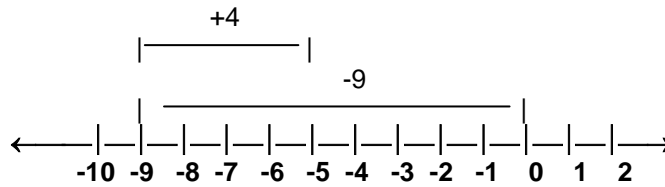
We can also add, using the number line.



Case 2: **Different Signs** → **SUBTRACT** the absolute value of the integers. Use the sign of the integer that has the larger absolute value.

- Example 2:**
- a) $-9 + 4 = |-9| - |4| = 9 - 4 = 5$, since -9 has the larger absolute value, the answer is -5.

Using a number line,



b) $5 + -2 = |5| - |-2| = 5 - 2 = 3$, since 5 has the larger absolute value, the answer is 3.

Using a number line,



If you have more than 2 integers to add together, you can either:

- (1) Work left to right, adding 2 at a time or,
- (2) Add the positives, add the negatives and then add those 2 answers.

Example 3: Add: $-7 + 10 + 12 + -8 + 3$

Using method (1)

$$\underline{-7 + 10} + 12 + -8 + 3$$

$$\underline{3 + 12} + -8 + 3$$

$$\underline{15 + -8} + 3$$

$$\underline{7 + 3}$$

$$10 .$$

Using method (2)

add the positives: $10 + 12 + 3 = 25$

add the negatives: $-7 + -8 = -15$

add those 2 answers: $25 + -15 = 10 .$

Now You Try (Section 2.2)

Calculate:

1) $18 + -5$

2) $-9 + -12$

3) $-23 + 10$

4) $27 + -24 + 12 + -6$

5) $-39 + -11 + -8 + 23$

(Answers to **Now You Try** (Section 2.2) are found on page 17.)

Section 3: Subtraction of Integers

When we subtract integers, we add the opposite.

Let a and b be any integers, then $a - b = a + (-b)$
--

Use the following steps when subtracting 2 integers:

- 1) Rewrite the 1st integer as it is given.
- 2) Change the subtraction sign to an addition sign.
- 3) Write the opposite of the 2nd number.
- 4) Use the rules for addition to find the answer.

Example 1:

- a) $12 - 27 = 12 + (-27) = -15$
↙ (different signs - subtract and keep the sign of the larger absolute value)
- b) $-12 - 27 = -12 + (-27) = -39$
↙ (same signs - add and keep the common sign)
- c) $-12 - (-27) = -12 + (+27) = 15$
↙ (different signs - subtract and keep the sign of the larger absolute value)
- d) $12 - (-27) = 12 + (+27) = 39$
↙ (same signs - add and keep the common sign)

If you have more than 2 integers to subtract, change all the subtractions to additions of the opposite and then use the rules for addition.

Example 2:

Calculate: $-5 - (-7) - 18 - (-6)$
 $= -5 + 7 + -18 + 6 = -23 + 13 = -10 .$

Now You Try (Section 3)

Calculate:

- | | |
|----------------------|---------------------------|
| 1) $-52 - 26$ | 2) $18 - 44$ |
| 3) $-12 - (-25)$ | 4) $36 - (-9)$ |
| 5) $-7 - 12 - (-23)$ | 6) $19 - 37 - 14 - (-38)$ |

(Answers to **Now You Try** (Section 3) are found on page 17.)

Section 4: Multiplication and Division of Integers

We have 2 different cases when multiplying or dividing 2 integers.

Case 1: Same Signs → MULTIPLY or DIVIDE as usual, answer will be POSITIVE.

Example 1:

a)	$7 \cdot 9 = 63$	c)	$\frac{35}{5} = 7$
b)	$-5 \cdot -3 = 15$	d)	$\frac{-48}{-6} = 8$

Case 2: Different Signs → MULTIPLY or DIVIDE as usual, answer will be NEGATIVE.

Example 2:

a)	$-7 \cdot 9 = -63$	d)	$\frac{48}{-6} = -8$
b)	$-5 \cdot 3 = -15$	e)	$\frac{-15}{0} = \text{undefined}$ (Recall that you cannot divide by 0.)
c)	$\frac{-35}{5} = -7$		

(Notice when multiplying or dividing two integers, it doesn't matter which number is larger, if one number is negative and the other is positive, the answer is negative.)

If you have more than 2 integers to multiply together you can either:

- (1) Work left to right, multiplying 2 at a time, or
- (2) Ignore the signs and multiply all the numbers, then count the # of negatives - if there are an even # of negatives, the answer is positive; if there are an odd # of negatives, the answer is negative.

Example 3: Multiply: $(-3)(-2)(-6)(-1)(4)$

Using method (1)

$$\underline{(-3)(-2)(-6)(-1)(4)}$$

$$\underline{6(-6)(-1)(4)}$$

$$\underline{-36(-1)(4)}$$

$$\underline{(36)(4)}$$

$$144 .$$

Using method (2)

$$(-3)(-2)(-6)(-1)(4)$$

$$(3)(2)(6)(1)(4) = 6 \cdot 6 \cdot 4 = 36 \cdot 4 = 144 .$$

Since there is an even number of negatives, the answer is positive.

Now You Try (Section 4)

Calculate:

1) $-5 \cdot 23$

2) $\frac{-78}{-2}$

3) $-36 \cdot -3$

4) $\frac{96}{-12}$

5) $(-4)(-3)(2)(-1)(5)$

6) $\frac{-8 \cdot 6}{0}$

(Answers to Now You Try (Section 4) are found on page 18.)

Section 5: Order of Operations

You know from previous courses that if two quantities are added, it does not make a difference which quantity is added to which. For example, $5 + 6 = 6 + 5$. This you will recall is the **commutative property of addition**. Even if you added more than two quantities it does not matter the order they are added. The same is true for multiplication. If you had to multiply $2 \cdot 4 \cdot 8$, you would always get 64 not matter what order the numbers were multiplied. What happens though when the operations are not all only addition or multiplication? For example, compute $2 + 6 \cdot 5$. You might do it from left to right and get 40 and a friend of yours might first multiply 6 and 5 and then add 2 to get an answer of 32. We cannot have this inconsistency, so it is important to have a set of rules (**order of operations**) to prevent different answers from occurring. The correct answer is 32. The order of operations tells us to multiply before we add.

The following is a summary of the order of operations.

ORDER OF OPERATIONS

If no parentheses are present do the following:

1. Evaluate all exponents, working from left to right.
2. Do any multiplications and divisions in the order in which they occur, reading from left to right.
3. Do any additions or subtractions in the order in which they occur, reading from left to right.

If parentheses are present, do the following:

1. First use the above steps within each pair of parentheses.
2. If the expression includes nested parentheses, evaluate the expression in the innermost set of parentheses first.

Note: Some people use the acronym "Please Excuse My Dear Aunt Sally" to remember the order of operations. Here the P stands for parentheses, E for exponent, M for multiplication, D for division, A for addition, and S for subtraction. Be careful to remember that multiplication and division are in the same step as are addition and subtraction.

Example 1: Evaluate $7 + 3 \cdot 2 - 18 \div 6$ without using a calculator.

Solution: No parentheses or exponents, so do multiplications and divisions from left to right in the order they appear,

$$\begin{aligned} 7 + 3 \cdot 2 - 18 \div 6 &= 7 + 6 - 3 \\ &\quad \quad \quad \uparrow \quad \uparrow \\ &\quad \quad \quad 3 \cdot 2 \quad 18 \div 6 \\ &= 13 - 3 && \text{Working left to right, add 7 and 6.} \\ &= 10 && \text{Subtract 3 from 13.} \end{aligned}$$

Example 2: Evaluate $6 + 3(8 - 3)$ without using a calculator.

Solution: According to the order of operations, we have to do what is inside the parentheses first:

$$\begin{aligned} 6 + 3(8 - 3) &= 6 + 3(5) && \text{Do inside the parentheses first.} \\ &= 6 + 15 && \text{Do multiplication before addition.} \\ &= 21 && \text{Add 6 and 15.} \end{aligned}$$

Example 3: Evaluate $6 - 4[8 + 3(7 - 3)]$ without using a calculator.

Solution: This expression has nested parentheses where the [] are used in the same way as parentheses. In a case like this we move to the innermost set of parentheses first and begin simplifying:

$$\begin{aligned} 6 - 4[8 + 3(7 - 3)] &= 6 - 4[8 + 3(4)] && (7-3); \text{ innermost grouping first.} \\ &= 6 - 4[8 + 12] && 3(4); \text{ Multiplication before addition.} \\ &= 6 - 4[20] && (8 + 12); \text{ Stay within [] until you get a single number.} \\ &= 6 - 80 && 4[20]; \text{ Multiplication before subtraction.} \\ &= - 74 && \text{Subtract 80 from 6.} \end{aligned}$$

Example 4: Evaluate $18 \div 3 \cdot 2 + 7 - 2 \cdot 4$ without using a calculator.

Solution: There are no parentheses or exponents, so we go from left to right and do all multiplications and divisions in the order they appear. In this case a division comes before a multiplication, so the division is done first.

$$\begin{aligned} 18 \div 3 \cdot 2 + 7 - 2 \cdot 4 &= 6 \cdot 2 + 7 - 2 \cdot 4 \\ &= 12 + 7 - 8 \\ &= 19 - 8 \\ &= 11 \end{aligned}$$

Example 5: Evaluate $9 \cdot 2^3 + 36 \div 3^2 - 8$ without using a calculator.

Solution: We do exponents first.

$$\begin{aligned} 9 \cdot 2^3 + 36 \div 3^2 - 8 &= 9 \cdot 8 + 36 \div 9 - 8 \\ &= 72 + 4 - 8 \\ &= 76 - 8 \\ &= 68 \end{aligned}$$

Now do multiplication and divisions from left to right.

Now do additions and subtractions from left to right.

Now You Try (Section 5)

Evaluate the following **without** using a calculator:

- 1) $3 + 4(3 + 12)$
- 2) $12 \div 6 \cdot 2 - 7(3 - 2)$
- 3) $4[6 - 4(2 - 7)]$

*(Answers to **Now You Try** (Section 5) are found on page 18.)*

Section 6: Exponents

You probably have some experience with exponents from previous courses. Just as multiplication can be thought of as a shorthand notation for repeated addition, $3 \cdot 4 = 4 + 4 + 4$, exponents are a shorthand notation for repeated multiplication, $2^3 = 2 \cdot 2 \cdot 2$. So **exponential notation** is just a shorthand system, but like any special notation you have to learn to read and use it correctly.

Exponential Notation

Let x be a real number and n be a positive integer.
The product of n of the x 's

$$\begin{array}{c} (x)(x)(x) \dots (x) \\ n \text{ times} \end{array}$$

is called

" x to the n th power"

or

" x to the n th"

or

" x raised to the n ."

We call n the **exponent** or "power" and x the **base**,
and write $(x)(x)(x) \dots (x)$ simply as x^n .
 n times

Now You Try (Section 6.2)

Evaluate the following without using a calculator.

1) $5 \cdot 3^4 + 16 \div 8 - 2^2$

2) $4 + 3(5 - (12 \div 6 \cdot 4 - 4^2))^2$

(Answers to Now You Try (Section 6.2) are found on page 18.)

Now let's look at evaluating expressions.

To evaluate expressions

- 1) Substitute the value for the variable.
- 2) Perform the operations given.

Example 1:

Evaluate:

a) $4x + 5y$ when $x = -2$ and $y = 4$

b) $6x^2$ when $x = -3$

c) $\frac{(x + y)^2}{12}$ when $x = 2$ and $y = -8$

Solution:

a) $4(-2) + 5(4)$ *Substitute -2 in for x and 4 in for y.*
 $-8 + 20$ *Multiply.*
 12 *Add.*

b) $6(-3)^2$ *Substitute -3 in for x.*
 $6 \cdot 9$ *Evaluate $(-3)^2 = (-3)(-3) = 9$.*
 54 *Multiply.*

$$c) \quad \frac{(2 + -8)^2}{12}$$

Substitute 2 in for x and -8 in for y.

$$\frac{(-6)^2}{12}$$

Evaluate what is in the parentheses.

$$\frac{36}{12}$$

Evaluate $(-6)^2 = (-6)(-6) = 36$.

$$3$$

Divide.

Now You Try (Section 6.3)

Evaluate:

1) $3x - 1$ when $x = 6$

2) $8x^2$ when $x = -4$

3) $\frac{(x - y)^3}{25}$ when $x = 2$ and $y = -3$

(Answers to **Now You Try** (Section 6.3) are found on page 18.)

Exercises for Integers

Do all the exercises on separate paper showing all your work.

1. Draw number line diagrams that illustrate each computation and state the answer in each case.

a) $8 + (-4)$

b) $-7 + 9$

c) $-5 - 5$

2. Write each of these subtractions as an addition.

a) $15 - 8$

b) $-5 - (-3)$

c) $-56 - (-64)$

d) $-33 - 18$

3. Perform the following computations.

a) $-12 - 14$

b) $75 - (43)$

c) $-23 - (-67)$

d) $5 + -8$

e) $-85 + -42$

f) $-12 - 12$

g) $-34 + 54$

h) $-34 - 54$

l) $15 + (-30) + 18 + -9$

j) $12 - (-21) - 34 - (-60)$

k) $25 - 54 + (-33)$

4. List these numbers in increasing order from least to greatest.

4, -3, 0, 1, -5, -56

5. Compute the following:

a) $18 \div -3$

b) $-6 \cdot -5$

c) $-6 + -14$

d) $-5 - -7$

e) $72 \cdot -2$

f) $-6(-14)$

g) $-48 \div 3$

h) $0 \div 18$

i) $-7 \div 0$

j) $(7)(-5)(2)(0)$

k) $(-5)(-2)(-4)(3)$

6. True or false

- a) The sum of a positive number and a negative number is negative.
- b) The difference of two negative numbers is positive.
- c) Dividing a positive integer by a negative integer, and multiplying the result by a negative integer gives a positive result.

7. Evaluate each of the following without using a calculator.

- a) $12 - 4 \div 2$
- b) $10 - 4(20 \div 4 \cdot 5 - 7 + 5)$
- c) $7[6 - 4(2 - 5)]$
- d) $3 - 6(3 \cdot 2 - 2)^2$
- e) $4 \cdot 5^3 - 20 \div 5 \cdot 4$
- f) $6^2 - 10 \div 2 \cdot 5^2$
- g) $5(3 + 4)^2$

8. Using exponents, write the following in simpler form:

- a) $(3)(3)(3)(3)(3)$
- b) $(6)(6)(6)(5)(5)$
- c) $a \cdot a \cdot a \cdot b \cdot b$

9. Name the base and the exponent in each of the following:

- a) $(3x)^2$
- b) $4x^3$
- c) $-4x^4$
- d) $(x + 6)^2$

10. Evaluate:

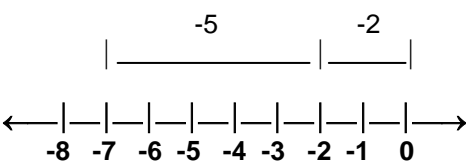
- a) $\frac{5(F - 32)}{9}$ when $F = 104$
- b) $\frac{5(F - 32)}{9}$ when $F = 0$
- c) $2(b - 5) + 7$ when $b = 16$
- d) $a^2 - 5a + 9$ when $a = -3$

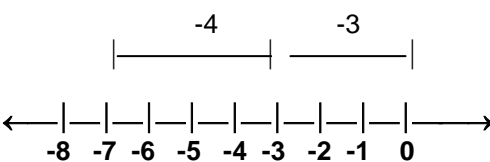
Answers to Now You Try

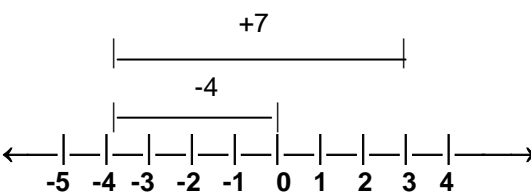
Section 1.1:

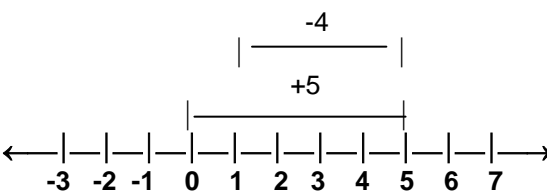
$$-7 < -5 < -1 < 0 < 2 < 6$$

Section 1.2:

1)  so, $-2 + -5 = -7$

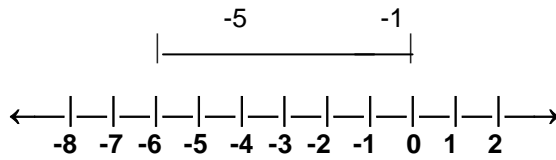
2)  so, $-3 + -4 = -7$

3)  so, $-4 + 7 = 3$

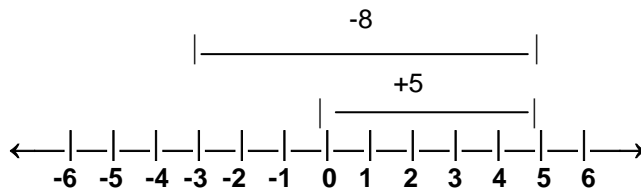
4)  so, $5 + -4 = 1$

Section 1.3:

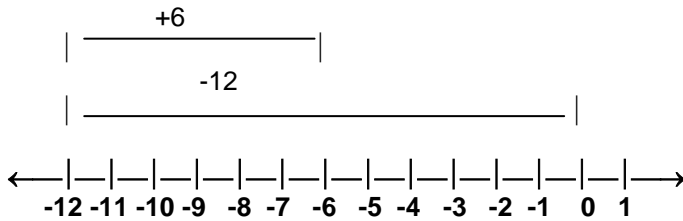
1) $-6 - 5 = -11$ since $-11 + 5 = -6$



2) $3 - 8 = -5$ since $-5 + 8 = 3$



3) $-6 - 6 = -12$ since $-12 + 6 = -6$



Section 2.1:

- 1) 7 2) 0 3) 24

Section 2.2:

- 1) 13 2) -21 3) -13 4) 9 5) -35

Section 3:

- 1) -78 2) -26 3) 13
4) 45 5) 4 6) 6

Section 4:

- 1) -115 2) 39 3) 108
4) -8 5) -120 6) $\frac{-48}{0} = \text{undefined}$

Section 5:

- 1) 63 2) -31 3) 104

Section 6.1:

- 1) -5 2) 5 (only the 5 is squared, the negative is not included.)
3) $x + y$ 4) x (only the x is cubed, not the 4.)

Section 6.2:

- 1) 403 2) 511

Section 6.3:

- 1) 17 2) 128 3) 5